A TRACTABLE FORMALISM FOR COMBINING RECTANGULAR CARDINAL RELATIONS WITH METRIC CONSTRAINTS

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Abstract: Knowledge representation and reasoning in real-world applications often require to integrate multiple aspects of space. In this paper, we focus our attention on the so-called Rectangular Cardinal Direction calculus for qualitative spatial reasoning on cardinal relations between rectangles whose sides are aligned to the axes of the plane. We first show how to extend a tractable fragment of such a calculus with metric constraints preserving tractability. Then, we illustrate how the resulting formalism makes it possible to represent available knowledge on directional relations between rectangles and to derive additional information about them, as well as to deal with metric constraints on the height/width of a rectangle or on the vertical/horizontal distance between rectangles.

1 INTRODUCTION

Qualitative spatial representation and reasoning play an important role in various areas of computer science such as, for instance, geographic information systems, spatial databases, document analysis, layout design, and image retrieval. Different aspects of space, such as direction, topology, size, and distance, which must be dealt with in a coherent way in many real-world applications, have been modeled by different formal systems (Broxvall, 2002; Condotta, 2000; Gerevini and Renz, 2002; Liu et al., 2009) (see (Cohn and Hazarika, 2001) for a survey). For practical reasons, a bidimensional space is commonly assumed, and spatial entities are represented by points, boxes, or polygons with a variety of shapes, depending on the required level of detail.

Information about spatial configurations is usually specified by constraint networks describing the allowed binary relations between pairs of spatial variables. The central problem in qualitative reasoning is *consistency checking*, which is the problem of deciding whether or not a network has a solution, that is, the problem of establishing whether or not there exists an assignment of domain values to variables that satisfies all constraints.

Cardinal relations are directional relationships that allow one to specify how spatial objects are placed relative to one another either by making use of a fixed reference system, e.g., to say that an object is to the "north" or "southwest" of another one in a geographic space, or, alternatively, by exploiting directions as "above" or "below and left" in a local space. Cardinal relations are of particular interest for geographic information systems, spatial databases, and image databases (Frank, 1996; Goyal, 2000; Papadias and Theodoridis, 1997; Skiadopoulos et al., 2005).

The most expressive formalism with cardinal relations between extended spatial objects is the *Cardinal Direction calculus*, *CD-calculus* for short (Goyal and Egenhofer, 2000; Liu et al., 2010; Skiadopoulos and Koubarakis, 2005). The consistency problem for the CD-calculus is NP-complete, and no tractable fragment of it has been identified so far, with the only exception of the fragment obtained by forbidding disjunctive relations (Skiadopoulos and Koubarakis, 2005). Such a restriction is a serious limitation when we have to deal with incomplete or indefinite information in spatial applications.

In (Navarrete and Sciavicco, 2006), the authors introduce a restricted version of the CD-calculus called *Rectangular Cardinal Direction calculus* (*RCD-calculus*), where cardinal relations are defined only between rectangles whose sides are parallel to the axes of the Euclidean plane. Rectangles of this type (boxes) can be seen as minimum bounding rectangles (MBRs) that enclose plane regions (the actual spatial objects). MBRs have been widely used in spatial databases (El-Geresy and Abdelmoty, 2001; Papadias and Theodoridis, 1997), in web-document analysis (Gatterbauer and Bohunsky, 2006), and in 2D-layout design, e.g., in architecture (Baykan and Fox, 1997). On the one hand, approximating regions by rectangles implies a loss of accuracy in the representation of the relative direction between regions; on the other hand, reasoning tasks become more efficient.

The RCD-calculus has a strong connection with the *Rectangle Algebra* (RA) (Balbiani et al., 1998), which can be viewed as a bidimensional extension of *Interval Algebra* (IA), the well-known temporal formalism for dealing with qualitative binary relations between time intervals (Allen, 1983). A tractable fragment of the RCD-calculus, named *convex RCD-calculus*, has been identified by Navarrete et al. in (Navarrete et al., 2011). It includes all basic relations and a large number of disjunctive relations, making it possible to represent and reason about indefinite information efficiently.

This paper aims at adding metric features to formalisms for qualitative spatial reasoning. Metric constraints between points over a dense linear order have been dealt with by the Temporal Constraint Satisfaction Problem formalism (TCSP) (Dechter et al., 1991). In such a formalism, one can constrain the distance between a pair of points to belong to a given set of intervals. If each constraint consists of one interval only, we get a tractable fragment of TCSP, called Simple Temporal Problem formalism (STP).

In the following, we propose a metric extension to the convex RCD-calculus that allows one to represent available knowledge on directional relations between rectangles and to derive additional information about them, as well as to deal with metric constraints on the height/width of a rectangle or on the vertical/horizontal distance between rectangles. We will show that the resulting formalism is expressive enough to capture various scenarios of practical interest and still computationally affordable.

The rest of the paper is organized as follows. In Section 2, we provide background knowledge on qualitative calculi and we shortly recall Interval Algebra and Rectangle Algebra. In Section 3, we introduce RCD-calculus and its convex fragment. In Section 4, we extend the convex RCD-calculus with metric constraints, and we devise a sound and complete polynomial algorithm for consistency checking. We conclude the section with a simple application example. Conclusions provide an assessment of the work and outline future research directions.

Figure 1: Basic relations of the Interval Algebra.

Relation	Symbol	Inverse	Illustration	Meaning
I before J	b	bi		$I^- < I^+ < J^- < J^+$
I overlaps J	o	oi		$I^- < J^- < I^+ < J^+$
I during J	d	di		$J^- < I^- < I^+ < J^+$
I meets J	m	mi		$I^- < I^+ = J^- < J^+$
I starts J	s	si		$I^- = J^- < I^+ < J^+$
I finishes J	f	fi		$J^- < I^- < I^+ = J^+$
I equals J	e	e		$I^- = J^- < I^+ = J^+$

2 PRELIMINARIES

In this section, we introduce basic notions and terminology.

Temporal knowledge, as well as spatial knowledge, is commonly represented in a qualitative calculus by means of a *qualitative network* consisting of a complete constraint-labeled digraph $N = (V,C)$, where $V = \{v_1,\ldots,v_n\}$ is a finite set of variables, interpreted over an infinite domain D, and the labeled edges in C specify the constraints describing qualitative spatial or temporal configurations. An edge from v_i to v_j labeled with R corresponds to the *constraint* $v_i R v_j$, where R denotes a binary relation over D which restricts the possible values for the pair of variables (v_i, v_j). The full set of relations of the calculus is usually taken as the powerset $2^{\mathcal{B}}$, where $\mathcal{B}$ is a finite set of *binary basic relations* that forms a partition of $D \times D$. Thus, a relation $R_{ij} \in 2^{\mathcal{B}}$ is of the form $R = \{r_1,\ldots,r_m\}$, where each r_i is a basic relation, and R represents the union of the basic relations it contains. If $m = 1$, we call R a *basic relation*; otherwise, we call it a *disjunctive relation*. A special case of disjunctive relation is the *universal relation*, denoted by '?', which contains all the basic relations. A *basic constraint* $v_i\{r\}v_j$ expresses definite knowledge about the values that the two variables v_i, v_j can take, while a *disjunctive constraint* $v_i\{r_1,\ldots,r_m\}v_j$ expresses indefinite or imprecise knowledge about these values. In particular, the *universal constraint* $v_i\,?\,v_j$ states that the relation between v_i an v_j is totally unknown. From a logical point of view, a disjunctive constraint $v_i\{r_1,\ldots,r_m\}v_j$ can be viewed as the logical disjunction $v_i\ \{r_1\}\ v_j \vee \cdots \vee v_i\ \{r_m\}\ v_j$.

An *instantiation* (or *interpretation*) of the constraints of a qualitative network N is a mapping ι representing an assignment of domain values to the variables of N. A constraint $v_i R v_j$ is said to be *satisfied* by an instantiation ι if the pair $(\iota(v_i), \iota(v_j))$ belongs to the binary relation represented by R. A *consistent* *instantiation*, or *solution*, of a network is an assignment of domain values to variables satisfying all the constraints. If such a solution exists, then the network is consistent, otherwise it is inconsistent.

The main reasoning task in qualitative reasoning is *consistency checking*, which amounts to deciding if a network is consistent. If all relations are considered, consistency checking is usually NP-hard. Hence, finding subsets of $2^{\mathcal{B}}$ for which consistency checking turns out to be polynomial (*tractable subsets*) is an important issue to address. Another common task in qualitative reasoning is computing the unique *minimal network* equivalent to a given one by determining, for each pair of variables, the *strongest* relation *(minimal relation*) entailed by the constraints of the network. It can be easily shown that each basic relation in a minimal network is *feasible*, i.e., it participates in some solution of the network.

To deal with these tasks, constraint propagation techniques are usually exploited. The most prominent method for constraint propagation is the *path-consistency algorithm*, PC-algorithm for short (Mackworth, 1977). Such an algorithm refines relations by successively applying the operation $R_{ij} \leftarrow R_{ij} \cap (R_{ik} \circ R_{kj})$ for every triple of variables (v_i, v_k, v_j), until a stable network is reached, where R_{ij}, R_{ik}, R_{kj} are the relations constraining the pair of variables $(v_i, v_j), (v_i, v_k), (v_k, v_j)$, respectively ($\circ$ stands for the composition of relations). If the empty relation is obtained during the process, then the input network is inconsistent; otherwise, we can conclude that the output network is *path consistent*, which does not necessarily imply that it is consistent. In some special cases, the PC-algorithm can be used to decide the consistency of a qualitative network and to get the minimal one.

2.1 Interval Algebra and Point Algebra

Allen's *Interval Algebra* (IA) allows one to model the relative position of two temporal intervals (Allen, 1983). An interval I is usually interpreted as a closed interval over the rational numbers $[I^-, I^+]$, whose *endpoints* I^- and I^+ satisfy the relation $I^- < I^+$. Let $\mathcal{B}_{ia}$ be the set of the thirteen *basic interval relations* capturing all possible ways to order the four endpoints of two intervals, usually denoted by the symbols $b, o, d, m, s, f, e, bi, oi, di, mi, si$, and fi. The semantics of basic IA-relations is defined in terms of ordering relations between the endpoints of the intervals, as shown in Figure 1. Notice that, given a basic relation r between two intervals I and J, the inverse relation ri is defined by simply exchanging the roles of I and J (see Figure 1). IA can be viewed as a constraint algebra defined by the power set $2^{\mathcal{B}_{ia}}$ and the operations of intersection, inverse ($^{-1}$), and composition ($\circ$) of relations.

IA subsumes *Point Algebra*, PA for short (Vilain and Kautz, 1986), a simpler qualitative calculus whose binary relations specify the relative position of pairs of time points. PA binary relations are $<, >, =$ (basic) and $\leq, \geq, \neq, ?$ (disjunctive), plus the empty relation. The endpoint relations defining an IA-relation (Figure 1) are basic relations of PA.

2.2 Rectangle Algebra

Rectangle Algebra (RA), proposed by Balbiani et al. (1998), is an extension of IA to a bidimensional space[1]. We assume here the domain of RA to consist of the set of rational rectangles whose sides are parallel to the axes of the Euclidean plane. To avoid a notational overload, with an abuse of notation, hereafter we will denote by a, b both rectangles in the domain of RA and constraint (rectangle) variables. A rectangle a is completely characterized by a pair of intervals (a_x, a_y), where a_x and a_y are the projections of a onto the x- and y-axis, respectively. We call $\mathcal{B}_{ra}$ the set of basic relations of RA, which is obtained by considering all possible pairs of basic IA-relations. Hence, a basic RA-relation r is denoted by a pair $r = (t, t')$ of basic IA-relations, representing the set of pairs of *rectangles* (a, b) such that $a\,(t, t')\,b$ holds if and only if, by definition, $a_x\, t\, b_x$ and $a_y\, t'\, b_y$ hold. Given a basic RA-relation $r = (t, t')$, let $t = \pi_x(r)$ and $t' = \pi_y(r)$ be the x- and y-projection of r, respectively.

Example 1. *Figure 2 shows a spatial realization of the basic RA-constraint $a\,\{(o, bi)\}\,b$. We have that $\pi_x(o, bi) = o$, $\pi_y(o, bi) = bi$, a_x overlaps b_x, and a_y is after b_y. The left endpoints of the intervals assigned to a_x and a_y (1 and 5.9, respectively) and their right endpoints (4.6 and 8, respectively) are the coordinates of the lower-left and upper-right vertices of the given instantiation of a, respectively. The same for b. Thus, the values assigned to the endpoints of the projections of a and b represent an assignment for a and b that satisfies the constraint $a\,\{(o, bi)\}\,b$.*

In the case of an arbitrary RA-relation $R \in 2^{\mathcal{B}_{ra}}$, the *projections of R* are defined as follows:

$$\pi_x(R) = \{\pi_x(r) \mid r \in R\} \quad \pi_y(R) = \{\pi_y(r) \mid r \in R\}.$$

Notice that, in general, $\pi_x(R) \times \pi_y(R)$ may be different from R or, equivalently, we may have $\pi_x(R_1) = \pi_x(R_2)$ and $\pi_y(R_1) = \pi_y(R_2)$ for some $R_1 \neq R_2$.

The mappings π_x and π_y can be generalized to RA-networks. We define the projections π_x and π_y

[1] An extension of RA to n-dimensional spaces can be found in (Balbiani et al., 2002).

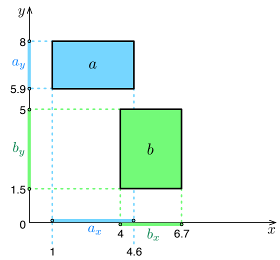
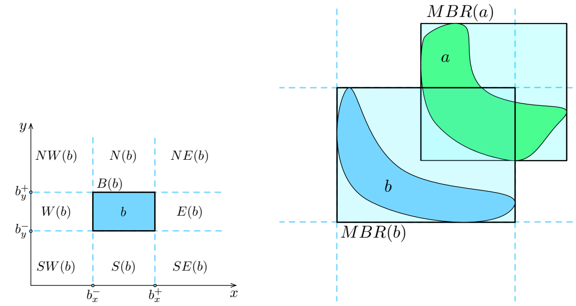


Figure 2: An instantiation of the RA-constraint $a\{(o,bi)\}b$. The corresponding RCD-relation is $a\,\{NW{:}N\}\,b$



(a) (b)

Figure 3: (a) Cardinal tiles with respect to rectangle b. (b) A possible instantiation of the RCD-constraint $a\,B{:}N{:}NE{:}E\,b$.

of an RA-network $N=(V,C)$ as the two IA-networks $\pi_x(N)=(V_x,C_x)$ and $\pi_y(N)=(V_y,C_y)$, where V_x,V_y are the sets of interval variables corresponding to the rectangle variables in V and the set of IA-constrains C_x (resp., C_y) is obtained by replacing each relation R_{ij} in C by $\pi_x(R_{ij})$ (resp., by $\pi_y(R_{ij})$).

2.3 Convex Subalgebras

The consistency problem for both IA and RA is known to be NP-complete. Several tractable fragments of both calculi have been identified in the literature. In this paper, we focus our attention on convex tractable subsets of IA (van Beek and Cohen, 1990) and RA (Balbiani et al., 1998), which consist of the set of *convex IA-relations* and *convex RA-relations*, respectively. Convex relations are those relations that can be equivalently expressed as a set of convex PA-constraints (all PA-relations except $\neq$ are allowed) between the endpoints of interval variables (convex IA-relations) or between the endpoints of the projections of rectangle variables (convex RA-relations) It is worth to mention that a convex RA-relation is equivalently characterized as a RA-relation which can be obtained as the Cartesian product of two convex IA-relations. A PC-algorithm can be used to solve both the consistency and the minimality problems in the convex fragments of PA, IA, and RA in $O(n^3)$, where n in the number of variables of the input network.

3 RECTANGULAR CARDINAL DIRECTION CALCULUS

The Rectangular Cardinal Direction calculus (RCD-calculus, for short) (Navarrete and Sciavicco, 2006; Navarrete et al., 2011) deals with cardinal direction relations between rectangles. Hence, its domain is the same as that of RA. Let b be a *reference* rectangle. We denote by b_x^- and b_x^+ (resp., b_y^- and b_y^+) the left and the right endpoint of the projection of b onto the x-axis (resp., y-axis), respectively. The straight lines $x=b_x^-$, $x=b_x^+$, $y=b_y^-$, $y=b_y^+$ divide the plane into nine tiles $\tau_i(b)$, with $1\leq i\leq 9$, as shown in Figure 3-(a), where τ_i is a *tile symbol* from the set $TS=\{B,S,SW,W,NW,N,NE,E,SE\}$, denoting the cardinal directions in the Bounds of, to the South of, to the SouthWest of, to the West of, to the NorthWest of, to the North of, to the NorthEast of, to the East of, and to the SouthEast of, respectively.

Definition 1. *A* basic rectangular cardinal relation *(basic RCD-relation) is denoted by a* tile string *$\tau_1{:}\tau_2{:}\ldots{:}\tau_k$, where $\tau_i\in TS$, for $1\leq i\leq k$, such that $a\,\tau_1{:}\tau_2{:}\ldots{:}\tau_k\,b$ holds iff for all $\tau_i\in\{\tau_1,\tau_2,\ldots,\tau_k\}$, $a^\circ\cap\tau_i(b)\neq\varnothing$, and for all $\tau_i\in TS\setminus\{\tau_1,\tau_2,\ldots,\tau_k\}$, $a^\circ\cap\tau_i(b)=\varnothing$, where a° is the interior of a. A* rectangular cardinal relation *(RCD-relation) is represented by a set $R=\{r_1,\ldots,r_m\}$, where each r_i is a basic RCD-relation.*

As usual, if R is a singleton, then it is a basic RCD-relation; otherwise, it is a disjunctive one.

The set $\mathcal{B}_{rcd}$ of basic RCD-relations consists of 36 elements (see Figure 4). Qualitative networks with labels in $2^{\mathcal{B}_{rcd}}$, as well as the consistency problem for such networks, are defined in the standard way.

The RCD-calculus can be viewed as a restricted version of the CD-calculus over the domain of regular regions (Goyal and Egenhofer, 2000; Liu et al., 2010; Skiadopoulos and Koubarakis, 2005), which includes all rectangles aligned to the axes. Let a,b denote regions. A cardinal relation is defined by considering the exact shape of a primary region a and the *minimum bounding rectangle* (MBR) of the reference region b, where MBR(b) is the smallest rectangle aligned to the axes of the plane that encloses b. There are 218 CD-relations over connected regions, that become 512 if we allow disconnected regions. Cardinal relationships between regions may be approximated by RCD-relations between their MBRs, with a possible loss of accuracy when the regions are non-convex or diagonal. The advantage of the RCD-calculus over the CD-calculus is its simplicity (only 36 basic relations), which leads to a better computational behavior, also when disjunctive relations are considered.

Example 2. *Figure 3-(b) shows a possible instantiation of the CD-constraint* $a\,B{:}N{:}E\,b$. *We indeed have that a lies partly in the bounds, partly to the north, and partly to the east of* $\textsc{mbr}(b)$. *Alternatively, the pair* $(\textsc{mbr}(a), \textsc{mbr}(b))$ *in Figure 3-(b) can be viewed as an instantiation of the RCD-constraint* $a\,B{:}N{:}NE{:}E,b$, *as it holds that* $\textsc{mbr}(a)\,B{:}N{:}NE{:}E\,\textsc{mbr}(a)$. *Notice that while the CD-constraint exactly specifies the direction of region a with respect to the minimum bounding rectangle of region b, the direction expressed by the RCD-constraint is just approximated, since a does not intersect the tile* $NE(b)$ $(= NE(\textsc{mbr}(b)))$, *that is, a does not lie partly to the northeast of* $\textsc{mbr}(b)$. *Notice also that, in general, a basic CD-constraint aRb alone does not provide definite information about the relative direction of pairs of regions. For that purpose, both aRb and* $bR'a$ *must be specified.*

3.1 RCD and RA

The relationships between RCD and RA have been systematically investigated in (Navarrete et al., 2011). For instance, consider the RCD-constraint $a\,\{NW{:}N\}\,b$. A possible instantiation of such a constraint is depicted in Figure 2. The very same pair of rectangles can be viewed as an instance of the RA-constraint $a\,\{(o, bi)\}\,b$ as well. However, there exists another possible instantiation of the constraint $a\,\{NW{:}N\}\,b$ that satisfies the RA-constraint $a\,\{(o, mi)\}\,b$. In general, for a given RCD-constraint there exist more than one corresponding RA-constraints, while for a given RA-constraint there exists exactly one corresponding RCD-constraint. This is due to the coarseness of RCD-relations with respect to RA-relations. As an example, RCD does not allow one to precisely state that two given rectangles are externally connected or strictly disconnected, or to constrain their sides to be (or to be not) vertically (resp., horizontally) aligned. As a general rule, given an RCD-relation, we can always determine the strongest RA-relation it implies. As an example, the strongest RA-relation implied by $NW{:}N$ is $\{fi, o\} \times \{mi, bi\}$. Notice that such an RA-relation, which is entailed by a basic RCD-relation, is not a basic RA-relation.

The weaker expressive power of RCD with respect RA is not necessarily a problem. As an example, if an application is interested in pure cardinal information only, the expressiveness of RCD-relations suffices. Moreover, the constraint language of the RCD-calculus is closer to the natural language than the one of the RA. For example, stating that "rectangle *a* lies partly to the northwest and partly to the north of *b*" ($a\,\{NW{:}N\}\,b$) is much more natural than stating that "the *x*-projection of *a* is overlapping or finished by the *x*-projection of *b*, and the *y*-projection of *b* is ..." ($a\{fi, o\} \times \{mi, bi\}b$).

Basic RCD-relation ↦ RA-relation (I)	Basic RCD-relation ↦ RA-relation (II)
B ↦ {d, s, f, e} × {d, s, f, e}	W:NW ↦ {m, b} × {si, oi}
S ↦ {d, s, f, e} × {m, b}	E:SE ↦ {mi, bi} × {fi, o}
N ↦ {d, s, f, e} × {mi, bi}	NE:E ↦ {mi, bi} × {si, oi}
E ↦ {mi, bi} × {d, s, f, e}	S:SW:SE ↦ {di} × {m, b}
W ↦ {m, b} × {d, s, f, e}	NW:N:NE ↦ {di} × {mi, bi}
NE ↦ {mi, bi} × {mi, bi}	B:W:E ↦ {di} × {d, s, f, e}
NW ↦ {m, b} × {mi, bi}	B:S:N ↦ {d, s, f, e} × {di}
SE ↦ {mi, bi} × {m, b}	SW:N:NW ↦ {m, b} × {di}
SW ↦ {m, b} × {m, b}	NE:E:SE ↦ {mi, bi} × {di}
S:SW ↦ {fi, o} × {m, b}	B:S:SW:W ↦ {o, fi} × {o, fi}
S:SE ↦ {si, oi} × {m, b}	B:W:NW:N ↦ {o, fi} × {si, oi}
NW:N ↦ {fi, o} × {mi, bi}	B:S:E:SE ↦ {si, oi} × {o, fi}
N:NE ↦ {si, oi} × {mi, bi}	B:N:NE:E ↦ {si, oi} × {si, oi}
B:W ↦ {fi, o} × {d, s, f, e}	B:S:SW:W:NW:N ↦ {o, fi} × {di}
B:E ↦ {si, oi} × {d, s, f, e}	B:S:N:NE:E:SE ↦ {si, oi} × {di}
B:S ↦ {d, s, f, e} × {fi, o}	B:S:SW:W:E:SE ↦ {di} × {fi, o}
B:N ↦ {d, s, f, e} × {si, oi}	B:W:NW:N:NE:E ↦ {di} × {si, oi}
W:SW ↦ {m, b} × {fi, o}	B:S:SW:W:NW:N:NE:E:SE ↦ {di} × {di}

Figure 4: Translation from basic RCD-relations to RA-relations via *toRA* mapping.

Figure 4 describes a translation function, called *toRA*, to map a basic RCD-relation into the strongest entailed *RA*-relation. This mapping can be extended to translate arbitrary relations, constraints, and networks of RCD-calculus to their counterparts in RA, preserving consistency. More precisely, given a disjunctive relation *R*, $toRA(R)$ is obtained as the union of the translation of the basic relations in *R*, while, given an RCD-network $N = (V, C)$, the corresponding RA-network $toRA(N)$ is obtained by replacing each relation R_{ij} in *C* by $toRA(R_{ij})$. As the following theorem states, to decide the consistency of an RCD-network *N*, one can compute the corresponding RA-network $toRA(N)$ and then apply any algorithm for deciding the consistency of RA-networks (Navarrete et al., 2011).

Theorem 1. *An RCD-network N is consistent if and only if the RA-network* $toRA(N)$ *is consistent.*

3.2 The Convex Fragment of RCD

In (Navarrete and Sciavicco, 2006), the authors prove that the consistency problem for the RCD-calculus is NP-complete, and they identify a tractable subset of RCD-relations. A larger tractable fragment of RCD-calculus, called *convex RCD-calculus*, has been identified in (Navarrete et al., 2011). Such a fragment consists of all and only the RCD-relations *R* whose translation $toRA(R)$ is a convex RA-relation (*convex*

Algorithm 3.1 the algorithm **con-cRCD**

Require: a convex RCD-network N
1: $N_r \leftarrow toRA(N)$;
2: $N_x \leftarrow \pi_x(N_r)$; $N_y \leftarrow \pi_y(N_r)$;
3: $N_x^P \leftarrow toPA(N_x)$; $N_y^P \leftarrow toPA(N_y)$;
4: If CSPAN(N_x^P) or CSPAN(N_y^P) returns an empty network, then return 'inconsistent'; otherwise, return 'consistent'.

RCD-relations). It is possible to show that there exist 400 such relations.

As we already pointed out, the convex subclasses of IA, PA, and RA are tractable and PC-algorithms can be used to decide their consistency. In particular, the following result holds for RA (Balbiani et al., 1998):

Theorem 2. *Let N be a convex RA-network. N is path-consistent (resp., consistent) iff its projections $\pi_x(N)$ and $\pi_y(N)$ are path-consistent (resp., consistent). Moreover, if N is path consistent, then it is consistent.*

Making use of the above results, polynomial-time algorithms to solve the consistency and the minimality problems for convex RCD-networks have been proposed in (Navarrete et al., 2011). In the following, we will exploit one of these algorithms, called **con-cRCD**, that solves the two PA-networks corresponding to a convex RCD-network. Such an algorithm can be summarized as follows. Let N be a convex RCD-network. First, it applies the mapping $toRA$ to get the convex RA-network N_r corresponding to N. Then, it computes the projections N_x and N_y of N_r. Next, it applies the mapping $toPA$ to translate the convex IA-networks N_x and N_y into two equivalent PA-networks N_x^P and N_y^P with convex relations between intervals endpoints. Such a mapping is based on the list of the convex IA-relations and of their translations to PA given in (van Beek and Cohen, 1990). Finally, the algorithm CSPAN (van Beek, 1992) is applied to decide the consistency of the two convex PA-networks in $O(n^2)$ (we assume that this algorithm returns an empty network in case the input network is inconsistent). It can be easily shown that such an algorithm runs in $O(n^2)$. Algorithm 3.1 provides a pseudocode encoding of **con-cRCD**.

4 CONVEX-METRIC RCD

In this section, we propose a tractable metric extension of the convex RCD-calculus, called *convex-metric RCD*, to represent and to reason with both qualitative cardinal constraints between rectangles and metric constraints on the distance between the endpoints of their projections.

4.1 STP

The main tool we use to deal with metric information in convex-metric RCD is the STP formalism, which was introduced in (Dechter et al., 1991) to process metric information about time points. More precisely, we use STP to elaborate information on the endpoints of MBR projections onto the Cartesian axes.

Formally, an STP is specified by a constraint network $S = (P, M)$, where P is a set of point variables, whose values range over a dense domain (we assume it to be $\mathbb{Q}$), and M is a set of *binary metric* constraints over P. A metric constraint $M_{ij} = [q, q']$ (open and semi-open intervals can be used), with $q, q' \in \mathbb{Q}$, on the distance between (the values of) $p_i, p_j \in P$ states that $p_j - p_i \in [q, q']$, or, equivalently, that $q \leq p_j - p_i \leq q'$. Hence, the constraint M_{ij} defines the set of possible values for the distance $p_j - p_i$. In the constraint graph associated to S, $M_{ij} = [q, q']$ is represented by an edge from p_i to p_j labeled by the rational interval $[q, q']$. *Unary metric* constraints restricting the domain of a point variable p_i can be encoded as binary constraints between p_i and a special starting-point variable with a fixed value, e.g., 0. The *universal constraint* is $]-\infty, +\infty[$. The operations of composition ($\circ$) and inverse ($^{-1}$) of metric constraints are computed by means of interval arithmetic, that is, $[q_1, q_2] \circ [q_3, q_4] = [q_1 + q_3, q_2 + q_4]$ and $[q_1, q_2]^{-1} = [-q_2, -q_1]$. Intersection of constraints (intervals) is defined as usual.

Assuming such an interpretation of the operations of composition, inverse, and intersection, Dechter et al. (1991) showed that any PC-algorithm can be exploited to compute the minimal STP equivalent to a given one, if any (if an inconsistency is detected, the algorithm returns an empty network). In the following, we will denote such an algorithm by $\mathbf{PC_{stp}}$. Making use of such a result, Meiri (1996) proposed a formalism to combine qualitative constraints between points and intervals with (possibly disjunctive) metric constraints between points (as in TCSP). An easy special case arises when only convex PA-constraints and STP-constraints are considered. Convex PA-constraints can be encoded as STP-constraints by means of the $toSTP$ translation function described in Table 1. The following result can be found in Meiri (1996):

Theorem 3. *Let N be a network with convex PA-constraints and STP-constraints. If N is path-consistent, then N is also consistent and its metric*

Table 1: Translation of convex PA-constraints to STP-constraints via the *toSTP* mapping.

Convex PA relation	STP constraint
$p_i < p_j$	$p_j - p_i \in]0, +\infty[$
$p_i \leq p_j$	$p_j - p_i \in [0, +\infty[$
$p_i = p_j$	$p_j - p_i \in [0, 0]$
$p_i > p_j$	$p_j - p_i \in]-\infty, 0[$
$p_i \geq p_j$	$p_j - p_i \in]-\infty, 0]$
$p_i \, ? \, p_j$	$p_j - p_i \in]-\infty, +\infty[$

constraints are minimal.

PC$_{\mathbf{stp}}$ can thus be used to decide the consistency of a network N satisfying the conditions of the above theorem. To this end, it suffices to encode PA-constraints into equivalent STP-constrains.

4.2 Integrating Convex RCD with STP

Combining RCD with STP makes it possible to express both directional constraints and metric constraints in a uniform framework. As an example, the resulting formalism allows one to constrain the position of a rectangle in the plane and to impose minimum and/or maximum values to the width/height of a given rectangle, or on the vertical/horizontal distances between the sides of two rectangles. Obviously, RCD-constraints and STP-constraints are not totally independent, that is, RCD-constraints entail some metric constraints and vice versa.

Example 3. *Let a and b be two rectangle. We can use the metric constraint* $0 < a_x^+ - a_x^- \leq 7$ *to state that the maximum width of a is* 7 *and, similarly, we can exploit the metric constraint* $2 \leq a_y^+ - a_y^-$ *to state that the minimum height of a is* 2 *(leaving the maximum height unbounded). We can also express distance constraints between the boundaries of a and b. We can constrain the horizontal distance between the right side of a and the left side of b to be at least* 3 *by means of the constraint* $3 \leq b_x^- - a_x^+$*, and the vertical distance between the upper side of a and the bottom side of b to be greater than or equal to* 0 *by means of the constraint* $0 \leq b_y^- - a_y^+$*. The two constraints together entail the basic RCD constraint* $a\,\{SW\}\,b$*. Finally, some metric constraints can be entailed by RCD ones. For instance, the convex relation* $a\,\{NW, N, NE, NW{:}N, NW{:}N{:}NE, N{:}NE\}\,b$ *implies that* $0 \leq a_y^- - b_y^+$.

If we allow one to combine arbitrary RCD-constraints with metric constrains, then checking the consistency of the resulting set of constraints turns out to be an NP-complete problem (the consistency problem for RCD-networks is already NP-complete). To preserve tractability, we restrict our attention to the combination of *convex* RCD-constraints with STP-constraints to establish the *convex-metric RCD* formalism.

Given a convex RCD-network $N_c = (V, C)$, we denote the sets of interval variables belonging to the projections $\pi_x(toRA(N_c))$ and $\pi_y(toRA(N_c))$ by V_x and V_y, respectively. Moreover, we denote by $P(V_x)$ and $P(V_y)$ the sets of point variables representing the endpoints of the interval variables in V_x and V_y, respectively. A convex-metric RCD-network is formally defined as follows.

Definition 2. *A convex-metric RCD-network (*cmRCD-network*) is an integrated qualitative and metric constraint network N consisting of three sub-networks* $(N_c, \mathcal{S}_x, \mathcal{S}_y)$*, where* $N_c = (V, C)$ *is a convex RCD-network, and* $\mathcal{S}_x = \big(P(V_x), M_x\big)$ *and* $\mathcal{S}_y = \big(P(V_y), M_y\big)$ *are two STPs.*

The convex-metric RCD formalism we propose subsumes the STP formalism and the convex RCD-calculus. Moreover, it also generalizes the convex fragment of the RA, since convex RA-relations are expressible as convex PA-relations and these relations can be, in turn, encoded into an STP.

Now, we provide an algorithm to solve the consistency problem for cmRCD that runs in $O(n^3)$[2]. First, we extend the translation mapping $toSTP$ of Table 1 to encode a convex PA-network N^P into an STP S by replacing each relation R_{ij} in the network N^P by $toSTP(R_{ij})$. By exploiting such a function, we can generalize the algorithm **con-cRCD** of Section 3.2 to deal with both RCD- and STP-constrains (Algorithm **con-cmRCD**). First, **con-cmRCD** computes the PA-networks N_x^P and N_y^P, and then, making use of information about convex RCD-relations encoded as PA-relations, it looks for possible inconsistencies between these constraints and the STP-constrains on the same variables given in S_x and S_y that can be detected at this stage. To this end, it translates the PA-network N_x^P (resp., N_y^P) into an STP-network by applying the function $toSTP$, and then it uses the function $intersect$ to compute the "intersection" between $toSTP(N_x^P)$ and S_x (resp., $toSTP(N_y^P)$ and S_y). This function simply intersects the intervals / metric constrains associated with the same pairs of variables in the two STPs. If an interval intersection produces an empty interval, then $intersect$ returns an empty network, and we can

[2]A similar combination of qualitative and quantitative networks is given by preconvex-augmented rectangle networks by Condotta (2000), that subsume *cmRCD-networks*. An $O(n^5)$ algorithm for checking the consistency of these networks has been devised by Condotta (Condotta, 2000). We exploit the trade-off between expressiveness and complexity to obtain a more efficient consistency checking algorithm.

Algorithm 4.1 the algorithm **con-cmRCD**

Require: a cmRCD-network $N = (N_c, \mathcal{S}_x, \mathcal{S}_y)$
1: $N_r \leftarrow toRA(N_c)$;
2: $N_x \leftarrow \pi_x(N_r)$, $N_y \leftarrow \pi_y(N_r)$;
3: $N_x^P \leftarrow toPA(N_x)$, $N_y^P \leftarrow toPA(N_y)$;
4: $xSTP \leftarrow intersect(toSTP(N_x^P), \mathcal{S}_x)$;
5: $ySTP \leftarrow intersect(toSTP(N_y^P), \mathcal{S}_y)$;
6: if $xSTP$ or $ySTP$ is empty, then return ‘inconsistent’;
7: $xSTP^{min} \leftarrow PC_{stp}(xSTP)$;
8: $ySTP^{min} \leftarrow PC_{stp}(ySTP)$;
9: If $xSTP^{min}$ or $ySTP^{min}$ is empty, then return ‘inconsistent’; otherwise, return ‘consistent’.

conclude that N is inconsistent. Otherwise, we apply the path-consistency algorithm to the two STPs computed at lines 4 and 5 independently. The following theorem proves that **con-cmRCD** is sound and complete.

Theorem 4. *Given a cmRCD-network $N = (N_c, \mathcal{S}_x, \mathcal{S}_y)$, the algorithm* **con-cmRCD** *returns ‘consistent’ if and only if N is consistent.*

Proof. We basically follow the steps of the algorithm. By Theorem 1, N_c is consistent if and only if N_r is consistent, and, by Theorem 2, N_r is consistent if and only if N_x and N_y are consistent (they can be checked independently). Next, N_x and N_y are consistent if and only if N_x^P and N_y^P are consistent, since there is no loss in information in the translations (van Beek and Cohen, 1990). The consistency of N_x^P and N_y^P could be checked by computing the corresponding STPs and by applying PC_{stp}. However, we cannot apply PC_{stp} directly to the STPs $toSTP(N_x^P)$ and $toSTP(N_x^P)$ since the metric constraints of S_x and S_y must be taken into account. Hence, we compute $intersect(toSTP(N_x^P), S_x)$ and $intersect(toSTP(N_x^P), \mathcal{S}_y)$. If one of them returns an empty network, then N is inconsistent. Otherwise, we independently apply PC_{stp} to $xSTP$ and $ySTP$. By Theorem 3, if one of the two applications of PC_{stp} returns an empty network, then N is inconsistent; otherwise, the path-consistent STPs $xSTP^{min}$ and $ySTP^{min}$ are consistent (and minimal), and thus N is consistent. □

Theorem 5. *The complexity of the algorithm* **con-cmRCD** *is $O(Rn^3)$, where n is the number of variables and R is the maximum range of the network.*

Proof. The translation via $toRA$, the generation of a projection of a network, the transformation of a IA-network into a RA-network via $toPA$ and the last two encodings via $toSTP$ require $O(n^2)$ steps, since there are $O(n^2)$ constraints and each constraint can be translated in constant time. The function $toPA$ introduces two variables for each interval variable, so $xSTP$ and $ySTP$ have $O(n)$ variables each. Finally, PC_{stp} runs in $O(Rn^3)$ time, so the overall complexity is $O(Rn^3)$ time, where R is the maximum range of the network (for more details about the complexity of achieving path-consistency for combined networks see (Meiri, 1996)). □

Once we have computed the path-consistent STPs $xSTP^{min}$ and $ySTP^{min}$ with algorithm **con-cmRCD**, we can build a *solution* to the convex-metric RCD-network N by computing a solution for the points in $xSTP$ and $ySTP$, since the assignment for point variables defines a consistent assignment for rectangle variables (see Example 1). To this end, the algorithm **STP-SOLUTION** by Gerevini and Cristani (1997) (Gerevini and Cristani, 1997) can be used.

To illustrate the expressive power of the convex-metric RCD-calculus and its potential applications, we show an example regarding the design of 2D-layouts.

Example 4. *Uncle Scrooge wants to buy a plot of land (p) to build a new money bin (m), an office (o), a house (h) and a swimming pool (s) for Huey, Dewey, and Louie. The surfaces of the buildings are supposed to be rectangular, with sides aligned to the sides of the plot, which also has a rectangular shape. During the feasibility study of the project, the following requirements arose: i) the vertical and horizontal distance between the boundaries of p and any building it contains must be at least* $100m$ *for reasons of privacy; ii) the surface area of m is* $70m \times 70m$*; iii) m must lie somewhere between the northwest zone and the northeast zone of h, and the same w.r.t. o; iv) the vertical distance between m and h (resp., o) must be at least* $100m$ *because Uncle Scrooge does not want to be disturbed too much by his employees; v) the surface area of h is* $100m \times 50m$*, while the surface area of o is* $30m \times 70m$*; vi) o must lie between the northeast zone and east zone of h; vii) the horizontal distance between o and h must be at least 60m and at most 80m so that Huey, Dewey, and Louie can play without disturbing their uncle’s workers; viii) s is an olympic-size swimming pool so its surface area has to be at least* $50m \times 25m$ *and at most* $100m \times 50m$*; ix) s must be situated between the southwest zone and southeast zone of h, and the same w.r.t. o; x) the vertical distance between s and h and between s and o must be at least 50m.*

Let us see how to represent the requirements of the above example with a cmRCD-network. The qualitative part of the network contains the following convex

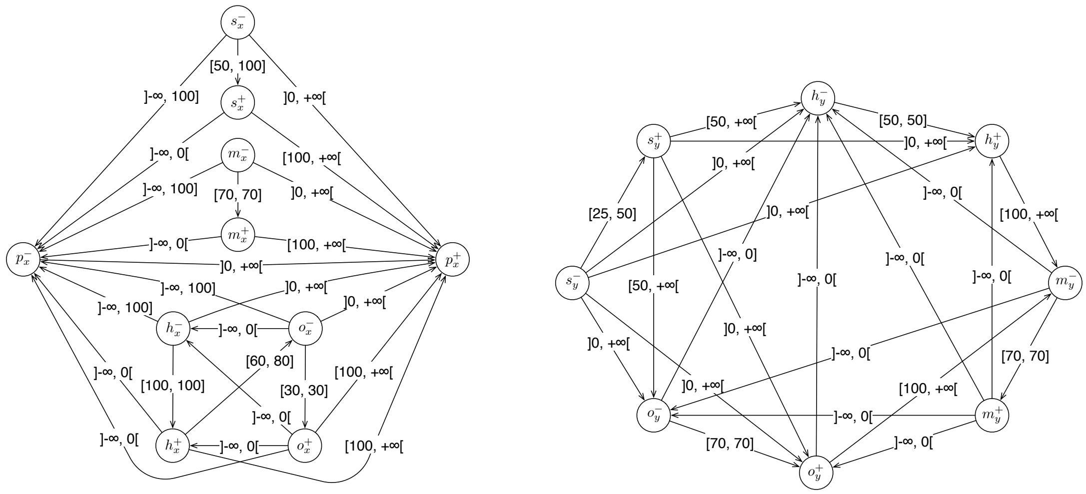


Figure 5: Graph representation of part of $xSTP$ and part of $ySTP$ of Example 4. For clarity, constraints involving p in $ySTP$ are omitted, as well as the universal constraint.

RCD-contraints between variables p, m, h, o, and s representing the plot and the buildings:

Implicit: "buildings must be inside the plot":
oBp, hBp, mBp, sBp;

iii) $m\{NW, N, NW{:}N, NW{:}N{:}NE, N{:}NE, NE\}h$,
$m\{NW, N, NW{:}N, NW{:}N{:}NE, N{:}NE, NE\}o$;

vi) $o\{NE, NE{:}E, E\}h$;

ix) $s\{SW, S, SW{:}S, SW{:}S{:}SE, S{:}SE, SE\}h$,
$s\{SW, S, SW{:}S, SW{:}S{:}SE, S{:}SE, SE\}o$;

The quantitative part of the network contains the following metric constraints forming two STPs:

i) for all buildings b:
$b_x^- - p_x^- \geq 100$, $p_x^+ - b_x^+ \geq 100$,
$b_y^- - p_y^- \geq 100$, $p_y^+ - b_y^+ \geq 100$

ii) $m_x^+ - m_x^- = 70$, $m_y^+ - m_y^- = 70$;

iv) $m_y^- - h_y^+ \geq 100$, $m_y^- - o_y^+ \geq 100$;

v) $h_x^+ - h_x^- = 100$, $h_y^+ - h_y^- = 50$,
$o_x^+ - o_x^- = 30$, $o_y^+ - o_y^- = 70$;

vii) $60 \leq o_x^- - h_x^+ \leq 80$;

viii) $50 \leq s_x^+ - s_x^- \leq 100$, $25 \leq s_y^+ - s_y^- \leq 50$

x) $h_y^- - s_y^+ \geq 50$, $o_y^- - s_y^+ \geq 50$.

By applying our consistency algorithm we can verify that it is possible to realize the building project of the example (the corresponding cmRCD-network is consistent). We can also determine the minimum area that the plot should have by using the minimal networks $xSTP^{min}$ and $ySTP^{min}$: in our example the minimum area of p is 390m×515m while the maximum area is unbounded. The STPs $xSTP$ and $ySTP$, computed by steps 4 and 5 of our algorithm, are sketched in Figure 5, while a solution of the problem is illustrated by Figure 6, showing the minimum feasible values for the point variables. To simplify, we suppose that the origin of the reference system is the lower-left vertex of the plot, since the plot encloses all the building and there is no constraint between the plot and the space around it.

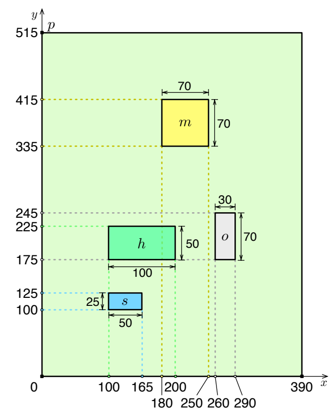


Figure 6: A solution to the cmRCD-network corresponding to Example 4.

5 CONCLUSIONS

In this paper, we have proposed a quite expressive, but tractable, metric extension of RCD (cmRCD), that integrates STP-constraints with convex RCD-constraints. cmRCD allows one to constrain the position of a rectangle in the plane, its width/height, and the vertical/horizontal distance between the sides of two rectangles, as well as to represent cardinal relations between rectangles. We have devised an $O(n^3)$ consistency-checking algorithm, and we have showed how a spatial realization of a network can be built.

As for future work, we plan to extend cmRCD with topological relations to improve its expressiveness (similar results can be found in (Gerevini and Renz, 2002; Liu et al., 2009)). The problem of identifying maximal tractable subsets of RCD is still open. It would be interesting to search for tractable classes (strictly) including the convex fragment.

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